

BOSONIZATION OF SUPERALGEBRA $U_q(\widehat{sl}(N|1))$ FOR AN ARBITRARY LEVEL

April 24, 2014

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Abstract

We give a bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$. The bosonization of level $k \in \mathbf{C}$ is completely different from those of level $k = 1$. From this bosonization, we induce the Wakimoto realization whose character coincides with those of the Verma module. We give the screening that commute with $U_q(\widehat{sl}(N|1))$. Using this screening, we propose the vertex operator that is the intertwiner among the Wakimoto realization and typical realization. We study non-vanishing property of the correlation function defined by a trace of the vertex operators.

1 Introduction

Bosonizations provide a powerful method to construct correlation function of exactly solvable models. We construct a bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ ($N \geq 2$) for an arbitrary level $k \in \mathbf{C}$ [1, 2]. For the special level $k = 1$, bosonizations have been constructed for the quantum affine algebra $U_q(g)$ in many cases $g = (ADE)^{(r)}$, $(BC)^{(1)}$, $G_2^{(1)}$, $\widehat{sl}(M|N)$, $osp(2|2)^{(2)}$ [3, 4, 5, 6, 7, 8, 9, 10]. Bosonizations of level $k \in \mathbf{C}$ are completely different from those of level $k = 1$. For an arbitrary level $k \in \mathbf{C}$ bosonizations have been studied only for $U_q(\widehat{sl}_N)$ [11, 12] and $U_q(\widehat{sl}(N|1))$ [1, 2]. Our construction is based on the ghost-boson system. We need more consideration to get the Wakimoto realization whose character coincides with those of the Verma module. Using ξ - η system we construct the Wakimoto realization [13, 14] from our level k bosonization. For an arbitrary level $k \neq -N + 1$ we construct the screening current that commutes with $U_q(\widehat{sl}(N|1))$ modulo total difference. By using Jackson integral and the screening current, we construct the screening that commute with $U_q(\widehat{sl}(N|1))$ [13, 15]. We propose the vertex operator that is the intertwiner among the Wakimoto realization and typical realization. By using the Gelfand-Zetlin basis, we have checked the intertwining property of the vertex operator for rank $N = 2, 3, 4$ [15]. We balance the background charge of the vertex operator by using the screening and propose the correlation function by a trace of them, which gives quantum and super generalization of Dotsenko-Fateev theory [16].

The paper is organized as follows. In section 2 we review bosonizations of $U_q(\widehat{sl}_2)$. In section 3 we construct a bosonization of $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$. We induce the Wakimoto realization by ξ - η system. In section 4 we construct the screening that commute with $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \neq -N + 1$. We propose the vertex operator and the correlation function.

2 Bosonization : Level $k = 1$ vs. Level $k \in \mathbf{C}$

In this section we review the bosonization of the quantum affine algebra $U_q(\widehat{sl}_2)$. The purpose of this section is to make readers understand that the bosonization of level $k \in \mathbf{C}$ is complete different from those of level $k = 1$. In what follows let q be a generic complex number $0 < |q| < 1$. We use the standard q -integer notation :

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

First we recall the definition of $U_q(\widehat{sl}_2)$. We recall the Drinfeld realization of the quantum affine algebra $U_q(\widehat{sl}_2)$.

Definition 2.1 [17] *The generators of the quantum affine algebra $U_q(\widehat{sl}_2)$ are $x_{i,n}^\pm$, h_m , h , c ($n \in \mathbf{Z}$, $m \in \mathbf{Z}_{\neq 0}$). Defining relations are*

$$\begin{aligned} c : \text{central}, [h, h_m] &= 0, \\ [h_m, h_n] &= \delta_{m+n,0} \frac{[2m]_q [cm]_q}{m}, \end{aligned}$$

$$\begin{aligned}
[h, x^\pm(z)] &= \pm 2x^\pm(z), \\
[h_m, x^\pm(z)] &= \pm \frac{[2m]_q}{m} q^{\mp \frac{c[m]}{2}} z^m x^\pm(z), \\
(z_1 - q^{\pm 2} z_2) x^\pm(z_1) x^\pm(z_2) &= (q^{\pm 2} z_1 - z_2) x^\pm(z_2) x^\pm(z_1), \\
[x^+(z_1), x^-(z_2)] &= \frac{1}{(q - q^{-1}) z_1 z_2} \\
&\times (\delta(q^{-c} z_1 / z_2) \psi^+(q^{\frac{c}{2}} z_2) - \delta(q^c z_1 / z_2) \psi^-(q^{-\frac{c}{2}} z_2)).
\end{aligned}$$

where we have used $\delta(z) = \sum_{n \in \mathbf{Z}} z^n$. We have set the generating function

$$\begin{aligned}
x^\pm(z) &= \sum_{n \in \mathbf{Z}} x_n^\pm z^{-n-1}, \\
\psi^\pm(q^{\pm \frac{c}{2}} z) &= q^{\pm h} e^{\pm(q - q^{-1}) \sum_{m > 0} h_{\pm m} z^{\mp m}}.
\end{aligned}$$

When the center c takes the complex number $c = k \in \mathbf{C}$, we call it the level k representation. We call the realization by the differential operators the bosonization. Frenkel-Jing [3] constructed the level $k = 1$ bosonization of the quantum affine algebra $U_q(g)$ for simply-laced $g = (ADE)^{(1)}$. Here we recall the level $k = 1$ bosonization of $U_q(\widehat{sl}_2)$. We introduce the boson a_n ($n \in \mathbf{Z}_{\neq 0}$) and the zero-mode operator ∂, α by

$$[a_m, a_n] = \frac{[2m]_q [m]_q}{m} \delta_{m+n, 0}, \quad [\partial, \alpha] = 2.$$

In what follows, in order to avoid divergences, we restrict ourselves to the Fock space of the bosons.

Theorem 2.2 [3] *A bosonization of the quantum affine algebra $U_q(\widehat{sl}_2)$ for the level $k = 1$ is given as follows.*

$$\begin{aligned}
c &= 1, \quad h = \partial, \quad h_n = a_n, \\
x^\pm(z) &=: e^{\mp \sum_{n \neq 0} \frac{a_n}{[n]_q} q^{\mp \frac{n}{2}} z^{-n} \pm (\alpha + \partial)} :.
\end{aligned}$$

We have used the normal ordering symbol ::

$$: a_k a_l := \begin{cases} a_k a_l & (k < 0), \\ a_l a_k & (k > 0), \end{cases} \quad : \alpha \partial := \partial \alpha := \alpha \partial.$$

Next we recall the level k bosonization of the quantum affine algebra $U_q(\widehat{sl}_2)$ [11]. We introduce the bosons and the zero-mode operator $a_n, b_n, c_n, Q_a, Q_b, Q_c$ ($n \in \mathbf{Z}$) as follows.

$$\begin{aligned}
[a_m, a_n] &= \delta_{m+n, 0} \frac{[2m]_q [(k+2)m]_q}{m}, \quad [\tilde{a}_0, Q_a] = 2(k+2), \\
[b_m, b_n] &= -\delta_{m+n, 0} \frac{[2m]_q [2m]_q}{m}, \quad [\tilde{b}_0, Q_b] = -4, \\
[c_m, c_n] &= \delta_{m+n, 0} \frac{[2m]_q [2m]_q}{m}, \quad [\tilde{c}_0, Q_c] = 4,
\end{aligned}$$

where $\tilde{a}_0 = \frac{q-q^{-1}}{2 \log q} a_0$, $\tilde{b}_0 = \frac{q-q^{-1}}{2 \log q} b_0$, $\tilde{c}_0 = \frac{q-q^{-1}}{2 \log q} c_0$. It is convenient to introduce the generating function $a(N|z; \alpha)$.

$$a(N|z; \alpha) = - \sum_{n \neq 0} \frac{a_n}{[Nn]_q} q^{|n|\alpha} z^{-n} + \frac{\tilde{a}_0}{N} \log z + \frac{Q_a}{N}.$$

In what follows, in order to avoid divergences, we restrict ourselves to the Fock space of the bosons.

Theorem 2.3 [11] *A bosonization of the quantum affine algebra $U_q(\widehat{sl}_2)$ for the level $k \in \mathbf{C}$ is given as follows.*

$$\begin{aligned}
c &= k \in \mathbf{C}, \quad h = a_0 + b_0, \\
h_m &= q^{2m-|m|} a_m + q^{(k+2)m - \frac{k+2}{2}|m|} b_m, \\
x^+(z) &= \frac{-1}{(q - q^{-1})z} \left(: e^{-b(2|q^{-k-2}z;1) - c(2|q^{-k-1}z;0)} : \right. \\
&\quad \left. - : e^{-b(2|q^{-k-2}z;1) - c(2|q^{-k-3}z;0)} : \right), \\
x^-(z) &= \frac{1}{(q - q^{-1})z} \left(: e^{a(k+2|q^kz, -\frac{k+2}{2}) - a(k+2|q^{-2}z; \frac{k+2}{2}) + b(2|z; -1) + c(2|q^{-1}z; 0)} : \right. \\
&\quad \left. - : e^{a(k+2|q^{-k-4}z; -\frac{k+2}{2}) - a(k+2|q^{-2}z; \frac{k+2}{2}) + b(2|q^{-2k-4}z; -1) + c(2|q^{-2k-3}z; 0)} : \right).
\end{aligned}$$

The level $k = 1$ bosonization is given by "monomial". The level $k \in \mathbf{C}$ bosonization is given by "sum". They are completely different.

3 Bosonization of Quantum Superalgebra $U_q(\widehat{sl}(N|1))$

In this section we study the bosonization of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$.

3.1 Quantum Superalgebra $U_q(\widehat{sl}(N|1))$

In this section we recall the definition of the quantum superalgebra $U_q(\widehat{sl}(N|1))$. We fix a generic complex number q such that $0 < |q| < 1$. The Cartan matrix $(A_{i,j})_{0 \leq i,j \leq N}$ of the affine Lie algebra $\widehat{sl}(N|1)$ is given by

$$A_{i,j} = (\nu_i + \nu_{i+1})\delta_{i,j} - \nu_i\delta_{i,j+1} - \nu_{i+1}\delta_{i+1,j}.$$

Here we set $\nu_1 = \dots = \nu_N = +, \nu_{N+1} = \nu_0 = -$. We introduce the orthonormal basis $\{\epsilon_i | i = 1, 2, \dots, N+1\}$ with the bilinear form, $(\epsilon_i | \epsilon_j) = \nu_i \delta_{i,j}$. Define $\bar{\epsilon}_i = \epsilon_i - \frac{\nu_i}{N-1} \sum_{j=1}^{N+1} \epsilon_j$. Note that $\sum_{j=1}^N \bar{\epsilon}_j = 0$. The classical simple roots $\bar{\alpha}_i$ and the classical fundamental weights $\bar{\Lambda}_i$ are defined by $\bar{\alpha}_i = \nu_i \epsilon_i - \nu_{i+1} \epsilon_{i+1}$, $\bar{\Lambda}_i = \sum_{j=1}^i \bar{\epsilon}_j$ ($1 \leq i \leq N$). Introduce the affine weight Λ_0 and the null root δ satisfying $(\Lambda_0 | \Lambda_0) = (\delta | \delta) = 0$, $(\Lambda_0 | \delta) = 1$, $(\Lambda_0 | \epsilon_i) = 0$, $(\delta | \epsilon_i) = 0$, ($1 \leq i \leq N$). The other affine weights and the affine roots are given by $\alpha_0 = \delta - \sum_{j=1}^N \bar{\alpha}_j$, $\alpha_i = \bar{\alpha}_i$, $\Lambda_i = \bar{\Lambda}_i + \Lambda_0$, ($1 \leq i \leq N$). Let $P = \oplus_{j=1}^N \mathbf{Z}\Lambda_j \oplus \mathbf{Z}\delta$ and $P^* = \oplus_{j=1}^N \mathbf{Z}h_j \oplus \mathbf{Z}d$ the affine $\widehat{sl}(N|1)$ weight lattice and its dual lattice, respectively.

Definition 3.1 [18] *The quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ are generated by the generators h_i, e_i, f_i ($0 \leq i \leq N$). The \mathbf{Z}_2 -grading of the generators are $|e_0| = |f_0| = |e_N| = |f_N| = 1$ and zero otherwise. The defining relations are given as follows.*

The Cartan-Kac relations : For $N \geq 2$, $0 \leq i, j \leq N$, the generators subject to the following relations.

$$[h_i, h_j] = 0, [h_i, e_j] = A_{i,j}e_j, [h_i, f_j] = -A_{i,j}f_j, [e_i, f_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}.$$

The Serre relations : For $N \geq 2$, the generators subject to the following relations for $1 \leq i \leq N-1$, $0 \leq j \leq N$ such that $|A_{i,j}| = 1$.

$$[e_i, [e_i, e_j]_{q^{-1}}]_q = 0, [f_i, [f_i, f_j]_{q^{-1}}]_q = 0.$$

For $N \geq 2$, the generators subject to the following relations for $0 \leq i, j \leq N$ such that $|A_{i,j}| = 0$.

$$[e_i, e_j] = 0, [f_i, f_j] = 0.$$

For $N \geq 3$, the Serre relations of fourth degree hold.

$$\begin{aligned} [e_N, [e_0, [e_N, e_{N-1}]_{q^{-1}}]_q] &= 0, [e_0, [e_1, [e_0, e_N]_q]_{q^{-1}}] = 0, \\ [f_N, [f_0, [f_N, f_{N-1}]_{q^{-1}}]_q] &= 0, [f_0, [f_1, [f_0, f_N]_q]_{q^{-1}}] = 0. \end{aligned}$$

For $N = 2$, the extra Serre relations of fifth degree hold.

$$\begin{aligned} [e_2, [e_0, [e_2, [e_0, e_1]_q]]_{q^{-1}} &= [e_0, [e_2, [e_0, [e_2, e_1]_q]]_{q^{-1}}, \\ [f_2, [f_0, [f_2, [f_0, f_1]_q]]_{q^{-1}} &= [f_0, [f_2, [f_0, [f_2, f_1]_q]]_{q^{-1}}. \end{aligned}$$

Here and throughout this paper, we use the notations

$$[X, Y]_\xi = XY - (-1)^{|X||Y|} \xi YX.$$

We write $[X, Y]_1$ as $[X, Y]$ for simplicity.

The quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ has the \mathbf{Z}_2 -graded Hopf-algebra structure. We take the following coproduct

$$\Delta(e_i) = e_i \otimes 1 + q^{h_i} \otimes e_i, \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i, \Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i,$$

and the antipode

$$S(e_i) = -q^{-h_i} e_i, S(f_i) = -f_i q^{h_i}, S(h_i) = -h_i.$$

The coproduct Δ satisfies an algebra automorphism $\Delta(XY) = \Delta(X)\Delta(Y)$ and the antipode S satisfies a \mathbf{Z}_2 -graded algebra anti-automorphism $S(XY) = (-1)^{|X||Y|} S(Y)S(X)$. The multiplication rule for the tensor product is \mathbf{Z}_2 -graded and is defined for homogeneous elements $X, Y, X', Y' \in U_q(\widehat{sl}(N|1))$ and $v \in V, w \in W$ by $X \otimes Y \cdot X' \otimes Y' = (-1)^{|Y||X'|} XX' \otimes YY'$ and $X \otimes Y \cdot v \otimes w = (-1)^{|Y||v|} Xv \otimes Yw$, which extends to inhomogeneous elements through linearity.

Definition 3.2 The quantum superalgebra $U_q(\widehat{sl}(N|1))$ is the subalgebra of $U_q(\widehat{sl}(N|1))$, that is generated by $e_1, e_2, \dots, e_N, f_1, f_2, \dots, f_N$, and h_1, h_2, \dots, h_N .

We recall the Drinfeld realization of $U_q(\widehat{sl}(N|1))$, that is convenient to construct bosonizations.

Definition 3.3 [18] *The generators of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ are $x_{i,n}^\pm$, $h_{i,m}$, h , c ($1 \leq i \leq N, n \in \mathbf{Z}, m \in \mathbf{Z}_{\neq 0}$). Defining relations are*

$$\begin{aligned}
c : \text{central}, \quad [h_i, h_{j,m}] &= 0, \\
[h_{i,m}, h_{j,n}] &= \frac{[A_{i,j}m]_q [cm]_q}{m} q^{-c|m|} \delta_{m+n,0}, \\
[h_i, x_j^\pm(z)] &= \pm A_{i,j} x_j^\pm(z), \\
[h_{i,m}, x_j^\pm(z)] &= \frac{[A_{i,j}m]_q}{m} q^{-c|m|} z^m x_j^\pm(z), \\
[h_{i,m}, x_j^\mp(z)] &= -\frac{[A_{i,j}m]_q}{m} z^m x_j^\mp(z), \\
(z_1 - q^{\pm A_{i,j}} z_2) x_i^\pm(z_1) x_j^\pm(z_2) &= (q^{\pm A_{j,i}} z_1 - z_2) x_j^\pm(z_2) x_i^\pm(z_1) \text{ for } |A_{i,j}| \neq 0, \\
[x_i^\pm(z_1), x_j^\pm(z_2)] &= 0 \text{ for } |A_{i,j}| = 0, \\
[x_i^+(z_1), x_j^-(z_2)] &= \frac{\delta_{i,j}}{(q - q^{-1}) z_1 z_2} (\delta(q^{-c} z_1/z_2) \psi_i^+(q^{\frac{c}{2}} z_2) - \delta(q^c z_1/z_2) \psi_i^-(q^{-\frac{c}{2}} z_2)), \\
(x_i^\pm(z_1) x_i^\pm(z_2) x_j^\pm(z) - (q + q^{-1}) x_i^\pm(z_1) x_j^\pm(z) x_i^\pm(z_2) + x_j^\pm(z) x_i^\pm(z_1) x_i^\pm(z_2)) \\
+ (z_1 \leftrightarrow z_2) &= 0 \text{ for } |A_{i,j}| = 1, i \neq N,
\end{aligned}$$

where we have used $\delta(z) = \sum_{m \in \mathbf{Z}} z^m$. Here we have used the generating function

$$\begin{aligned}
x_j^\pm(z) &= \sum_{m \in \mathbf{Z}} x_{j,m}^\pm z^{-m-1}, \\
\psi_i^\pm(q^{\pm \frac{c}{2}} z) &= q^{\pm h_i} e^{\pm(q - q^{-1}) \sum_{m > 0} h_{i,\pm m} z^{\mp m}}.
\end{aligned}$$

The relation between two definitions of $U_q(\widehat{sl}(N|1))$ are given by

$$\begin{aligned}
h_0 &= c - (h_1 + \cdots + h_N), \quad e_i = x_{i,0}^+, \quad f_i = x_{i,0}^- \text{ for } 1 \leq i \leq N, \\
e_0 &= (-1)[x_{N,0}^- \cdots, [x_{3,0}^-, [x_{2,0}^-, x_{1,1}^-]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}} q^{-h_1 - h_2 - \cdots - h_N}, \\
f_0 &= q^{h_1 + h_2 + \cdots + h_N} [\cdots [[x_{1,-1}^+, x_{2,0}^+]_q, x_{3,0}^+]_q, \cdots x_{N,0}^+]_q.
\end{aligned}$$

For instance we have the coproduct as follows.

$$\begin{aligned}
\Delta(h_{i,m}) &= h_{i,m} \otimes q^{\frac{cm}{2}} + q^{\frac{3cm}{2}} \otimes h_{i,m} \quad (m > 0), \\
\Delta(h_{i,-m}) &= h_{i,-m} \otimes q^{-\frac{3cm}{2}} + q^{-\frac{cm}{2}} \otimes h_{i,-m} \quad (m > 0).
\end{aligned}$$

3.2 Bosonization

In this section we construct bosonizations of quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$ [2]. We introduce the bosons and the zero-mode operators a_m^j, Q_a^j ($m \in \mathbf{Z}, 1 \leq j \leq N$), $b_m^{i,j}, Q_b^{i,j}$ ($m \in \mathbf{Z}, 1 \leq i < j \leq N+1$), $c_m^{i,j}, Q_c^{i,j}$ ($m \in \mathbf{Z}, 1 \leq i < j \leq N$) which satisfy

$$\begin{aligned}
[a_m^i, a_n^j] &= \frac{[(k + N - 1)m]_q [A_{i,j}m]_q}{m} \delta_{m+n,0}, \quad [a_0^i, Q_a^j] = (k + N - 1) A_{i,j}, \\
[b_m^{i,j}, b_n^{i',j'}] &= -\nu_i \nu_j \frac{[m]_q^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \quad [b_0^{i,j}, Q_b^{i',j'}] = -\nu_i \nu_j \delta_{i,i'} \delta_{j,j'},
\end{aligned}$$

$$\begin{aligned}
[c_m^{i,j}, c_n^{i',j'}] &= \frac{[m]_q^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \quad [c_0^{i,j}, Q_c^{i',j'}] = \delta_{i,i'} \delta_{j,j'}, \\
[Q_b^{i,j}, Q_b^{i',j'}] &= \delta_{j,N+1} \delta_{j',N+1} \pi \sqrt{-1} \quad (i,j) \neq (i',j').
\end{aligned}$$

Other commutation relations are zero. In what follows we use the standard symbol of the normal orderings \vdots . It is convenient to introduce the generating function $b^{i,j}(z)$, $c^{i,j}(z)$, $b_\pm^{i,j}(z)$, $a_\pm^j(z)$ and $\left(\frac{\gamma_1}{\beta_1} \cdots \frac{\gamma_r}{\beta_r} a^i\right)(z|\alpha)$ given by

$$\begin{aligned}
b^{i,j}(z) &= - \sum_{m \neq 0} \frac{b_m^{i,j}}{[m]_q} z^{-m} + Q_b^{i,j} + b_0^{i,j} \log z, \\
c^{i,j}(z) &= - \sum_{m \neq 0} \frac{c_m^{i,j}}{[m]_q} z^{-m} + Q_c^{i,j} + c_0^{i,j} \log z, \\
b_\pm^{i,j}(z) &= \pm(q - q^{-1}) \sum_{\pm m > 0} b_m^{i,j} z^{-m} \pm b_0^{i,j} \log q, \\
a_\pm^j(z) &= \pm(q - q^{-1}) \sum_{\pm m > 0} a_m^j z^{-m} \pm a_0^j \log q, \\
\left(\frac{\gamma_1}{\beta_1} \cdots \frac{\gamma_r}{\beta_r} a^i\right)(z|\alpha) &= - \sum_{m \neq 0} \frac{[\gamma_1 m]_q \cdots [\gamma_r m]_q}{[\beta_1 m]_q \cdots [\beta_r m]_q} \frac{a_m^i}{[m]_q} q^{-\alpha|m|} z^{-m} \\
&\quad + \frac{\gamma_1 \cdots \gamma_r}{\beta_1 \cdots \beta_r} (Q_a^i + a_0^i \log z).
\end{aligned}$$

In order to avoid divergence we work on the Fock space defined below. We introduce the vacuum state $|0\rangle \neq 0$ of the boson Fock space by

$$a_m^i |0\rangle = b_m^{i,j} |0\rangle = c_m^{i,j} |0\rangle = 0 \quad (m \geq 0).$$

For $p_a^i \in \mathbf{C}$ ($1 \leq i \leq N$), $p_b^{i,j} \in \mathbf{C}$ ($1 \leq i < j \leq N+1$), $p_c^{i,j} \in \mathbf{C}$ ($1 \leq i < j \leq N$), we set

$$\begin{aligned}
|p_a, p_b, p_c\rangle &= e^{\sum_{i,j=1}^N \frac{\text{Min}(i,j)(N-1-\text{Max}(i,j))}{(N-1)(N-1)} p_a^i Q_a^j} \\
&\quad \times e^{-\sum_{1 \leq i < j \leq N+1} p_b^{i,j} Q_b^{i,j} + \sum_{1 \leq i < j \leq N} p_c^{i,j} Q_c^{i,j}} |0\rangle.
\end{aligned}$$

It satisfies

$$\begin{aligned}
a_0^i |p_a, p_b, p_c\rangle &= p_a^i |p_a, p_b, p_c\rangle, \\
b_0^{i,j} |p_a, p_b, p_c\rangle &= p_b^{i,j} |p_a, p_b, p_c\rangle, \quad c_0^{i,j} |p_a, p_b, p_c\rangle = p_c^{i,j} |p_a, p_b, p_c\rangle.
\end{aligned}$$

The boson Fock space $F(p_a, p_b, p_c)$ is generated by the bosons $a_m^i, b_m^{i,j}, c_m^{i,j}$ on the vector $|p_a, p_b, p_c\rangle$. We set the space $F(p_a)$ by

$$F(p_a) = \bigoplus_{\substack{p_b^{i,j} = -p_c^{i,j} \in \mathbf{Z} \quad (1 \leq i < j \leq N) \\ p_b^{i,N+1} \in \mathbf{Z} \quad (1 \leq i \leq N)}} F(p_a, p_b, p_c).$$

We impose the restriction $p_b^{i,j} = -p_c^{i,j} \in \mathbf{Z}$ ($1 \leq i < j \leq N$). We construct a bosonization on the space $F(p_a)$.

Theorem 3.4 [2] *A bosonization of the quantum superalgebra $U_q(\widehat{\mathfrak{sl}}(N|1))$ for an arbitrary level $k \in \mathbf{C}$*

is given as follows.

$$\begin{aligned}
c &= k \in \mathbf{C}, \\
h_i &= a_0^i + \sum_{l=1}^i (b_0^{l,i+1} - b_0^{l,i}) + \sum_{l=i+1}^N (b_0^{i,l} - b_0^{i+1,l}) + b_0^{i,N+1} - b_0^{i+1,N+1}, \\
h_N &= a_0^N - \sum_{l=1}^{N-1} (b_0^{l,N} + b_0^{l,N+1}), \\
h_{i,m} &= q^{-\frac{N-1}{2}|m|} a_m^i + \sum_{l=1}^i (q^{-(\frac{k}{2}+l-1)|m|} b_m^{l,i+1} - q^{-(\frac{k}{2}+l)|m|} b_m^{l,i}) \\
&\quad + \sum_{l=i+1}^N (q^{-(\frac{k}{2}+l)|m|} b_m^{i,l} - q^{-(\frac{k}{2}+l-1)|m|} b_m^{i+1,l}) \\
&\quad + q^{-(\frac{k}{2}+N)|m|} b_m^{i,N+1} - q^{-(\frac{k}{2}+N-1)|m|} b_m^{i+1,N+1}, \\
h_{N,m} &= q^{-\frac{N-1}{2}|m|} a_m^N - \sum_{l=1}^{N-1} (q^{-(\frac{k}{2}+l)|m|} b_m^{l,N} + q^{-(\frac{k}{2}+l)|m|} b_m^{l,N+1}), \\
x_i^+(z) &= \frac{1}{(q - q^{-1})z} : \sum_{j=1}^i e^{(b+c)^{j,i}(q^{j-1}z) + \sum_{l=1}^{j-1} (b_+^{l,i+1}(q^{l-1}z) - b_+^{l,i}(q^l z))} \times \\
&\quad \times \left\{ e^{b_+^{j,i+1}(q^{j-1}z) - (b+c)^{j,i+1}(q^j z)} - e^{b_-^{j,i+1}(q^{j-1}z) - (b+c)^{j,i+1}(q^{j-2}z)} \right\} :, \\
x_N^+(z) &= : \sum_{j=1}^N e^{(b+c)^{j,N}(q^{j-1}z) + b^{j,N+1}(q^{j-1}z) - \sum_{l=1}^{j-1} (b_+^{l,N+1}(q^l z) + b_+^{l,N}(q^l z))} :, \\
x_i^-(z) &= q^{k+N-1} : e^{a_+^i(q^{\frac{k+N-1}{2}}z) - b^{i,N+1}(q^{k+N-1}z) - b_+^{i+1,N+1}(q^{k+N-1}z) + b^{i+1,N+1}(q^{k+N}z)} : \\
&\quad + \frac{1}{(q - q^{-1})z} : \left\{ \sum_{j=1}^{i-1} e^{a_-^i(q^{-\frac{k+N-1}{2}}z) + (b+c)^{j,i+1}(q^{-k-j}z) + b_-^{i,n+1}(q^{-k-n}z) - b_-^{i+1,n+1}(q^{-k-n+1}z)} \right. \\
&\quad \times e^{\sum_{l=j+1}^i (b_-^{l,i+1}(q^{-k-l+1}z) - b_-^{l,i}(q^{-k-l}z)) + \sum_{l=i+1}^N (b_-^{i,l}(q^{-k-l}z) - b_-^{i+1,l}(q^{-k-l+1}z))} \\
&\quad \times \left(e^{-b_-^{j,i}(q^{-k-j}z) - (b+c)^{j,i}(q^{-k-j+1}z)} - e^{-b_+^{j,i}(q^{-k-j}z) - (b+c)^{j,i}(q^{-k-j-1}z)} \right) \\
&\quad + e^{a_-^i(q^{-\frac{k+N-1}{2}}z) + (b+c)^{i,i+1}(q^{-k-i}z)} \\
&\quad \times e^{\sum_{l=i+1}^N (b_-^{i,l}(q^{-k-l}z) - b_-^{i+1,l}(q^{-k-l+1}z)) + b_-^{i,N+1}(q^{-k-N}z) - b_-^{i+1,N+1}(q^{-k-N+1}z)} \\
&\quad - e^{a_+^i(q^{\frac{k+N-1}{2}}z) + (b+c)^{i,i+1}(q^{k+i}z)} \\
&\quad \times e^{\sum_{l=i+1}^N (b_+^{i,l}(q^{k+l}z) - b_+^{i+1,l}(q^{k+l-1}z)) + b_+^{i,N+1}(q^{k+N}z) - b_+^{i+1,N+1}(q^{k+N-1}z)} \\
&\quad - \sum_{j=i+1}^{N-1} e^{a_+^i(q^{\frac{k+N-1}{2}}z) + (b+c)^{i,j+1}(q^{k+j}z)} \\
&\quad \times e^{b_+^{i,N+1}(q^{k+N}z) - b_+^{i+1,N+1}(q^{k+N-1}z) + \sum_{l=j+1}^N (b_+^{i,l}(q^{k+l}z) - b_+^{i+1,l}(q^{k+l-1}z))} \\
&\quad \times \left. \left(e^{b_+^{i+1,j+1}(q^{k+j}z) - (b+c)^{i+1,j+1}(q^{k+j+1}z)} - e^{b_-^{i+1,j+1}(q^{k+j}z) - (b+c)^{i+1,j+1}(q^{k+j-1}z)} \right) \right\} :. \\
x_N^-(z) &= \frac{1}{(q - q^{-1})z} : \left\{ \sum_{j=1}^{N-1} e^{a_-^N(q^{-\frac{k+N-1}{2}}z) - b_+^{j,N+1}(q^{-k-j}z) - b_-^{j,N+1}(q^{-k-j-1}z)} \right. \\
&\quad \times e^{-\sum_{l=j+1}^{N-1} (b_-^{l,N}(q^{-k-l}z) + b_-^{l,N+1}(q^{-k-l}z))}
\end{aligned}$$

$$\begin{aligned}
& \times q^{j-1} \left(e^{-b_+^{j,N}(q^{-k-j}z) - (b+c)^{j,N}(q^{-k-j-1}z)} - e^{-b_-^{j,N}(q^{-k-j}z) - (b+c)^{j,N}(q^{-k-j+1}z)} \right) : \\
& + q^{N-1} : \left(e^{a_+^N(q^{\frac{k+N-1}{2}}z) - b^{N,N+1}(q^{k+N-1}z)} - e^{a_-^N(q^{-\frac{k+N-1}{2}}z) - b^{N,N+1}(q^{-k-N+1}z)} \right) : .
\end{aligned}$$

3.3 Replacement from $U_q(sl(N|1))$ to $U_q(\widehat{sl}(N|1))$

In this section we study the relation between $U_q(sl(N|1))$ and $U_q(\widehat{sl}(N|1))$. Let us recall the Heisenberg realization of quantum superalgebra $U_q(sl(N|1))$ [1]. We introduce the coordinates $x_{i,j}$, ($1 \leq i < j \leq N+1$) by

$$x_{i,j} = \begin{cases} z_{i,j} & (1 \leq i < j \leq N), \\ \theta_{i,j} & (1 \leq i \leq N, j = N+1). \end{cases} \quad (3.1)$$

Here $z_{i,j}$ are complex variables and $\theta_{i,N+1}$ are the Grassmann odd variables that satisfy $\theta_{i,N+1}\theta_{i,N+1} = 0$ and $\theta_{i,N+1}\theta_{j,N+1} = -\theta_{j,N+1}\theta_{i,N+1}$, ($i \neq j$). We introduce the differential operators $\vartheta_{i,j} = x_{i,j} \frac{\partial}{\partial x_{i,j}}$, ($1 \leq i < j \leq N+1$).

Theorem 3.5 [1] *We fix parameters $\lambda_i \in \mathbf{C}$ ($1 \leq i \leq N$). The Heisenberg realization of $U_q(sl(N|1))$ is given as follows.*

$$\begin{aligned}
h_i &= \sum_{j=1}^{i-1} (\nu_i \vartheta_{j,i} - \nu_{i+1} \vartheta_{j,i+1}) + \lambda_i - (\nu_i + \nu_{i+1}) \vartheta_{i,i+1} + \sum_{j=i+1}^N (\nu_{i+1} \vartheta_{i+1,j+1} - \nu_i \vartheta_{i,j+1}), \\
e_i &= \sum_{j=1}^i \frac{x_{j,i}}{x_{j,i+1}} [\vartheta_{j,i+1}]_q q^{\sum_{l=1}^{j-1} (\nu_i \vartheta_{l,i} - \nu_{i+1} \vartheta_{l,i+1})}, \\
f_i &= \sum_{j=1}^{i-1} \nu_i \frac{x_{j,i+1}}{x_{j,i}} [\vartheta_{j,i}]_q q^{\sum_{l=j+1}^{i-1} (\nu_{i+1} \vartheta_{l,i+1} - \nu_i \vartheta_{l,i}) - \lambda_i + (\nu_i + \nu_{i+1}) \vartheta_{i,i+1} + \sum_{l=i+2}^{N+1} (\nu_i \vartheta_{i,l} - \nu_{i+1} \vartheta_{i+1,l})} \\
&+ x_{i,i+1} \left[\lambda_i - \nu_i \vartheta_{i,i+1} - \sum_{l=i+2}^{N+1} (\nu_i \vartheta_{i,l} - \nu_{i+1} \vartheta_{i+1,l}) \right]_q \\
&- \sum_{j=i+1}^N \nu_{i+1} \frac{x_{j,i+1}}{x_{i+1,j+1}} [\vartheta_{i+1,j+1}]_q q^{\lambda_i + \sum_{l=j+1}^{N+1} (\nu_{i+1} \vartheta_{i+1,l} - \nu_i \vartheta_{i,l})}.
\end{aligned}$$

Here we read $x_{i,i} = 1$ and, for Grassmann odd variables $x_{i,j}$, the expression $\frac{1}{x_{i,j}}$ stands for the derivative $\frac{1}{x_{i,j}} = \frac{\partial}{\partial x_{i,j}}$.

We study how to recover the bosonization of the affine superalgebra $U_q(\widehat{sl}(N|1))$ from the Heisenberg realization of $U_q(sl(N|1))$. We make the following replacement with suitable argument.

$$\begin{aligned}
\vartheta_{i,j} &\rightarrow -b_{\pm}^{i,j}(z)/\log q \quad (1 \leq i < j \leq N+1), \\
[\vartheta_{i,j}]_q &\rightarrow \begin{cases} \frac{e^{\pm b_+^{i,j}(z)} - e^{\pm b_-^{i,j}(z)}}{(q - q^{-1})z} & (j \neq N+1), \\ 1 & (j = N+1). \end{cases}
\end{aligned}$$

$$\begin{aligned}
x_{i,j} &\rightarrow \begin{cases} : e^{(b+c)^{i,j}(z)} : & (j \neq N+1), \\ : e^{-b^{i,j}(z)} : \text{ or } : e^{-b_{\pm}^{i,j}(q^{\pm 1}z)-b^{i,j}(z)} : & (j = N+1). \end{cases} \\
\lambda_i &\rightarrow a_{\pm}^i(z)/\log q \quad (1 \leq i \leq N), \\
[\lambda_i]_q &\rightarrow \frac{e^{\pm a_+^i(z)} - e^{\pm a_-^i(z)}}{(q - q^{-1})z} \quad (1 \leq i \leq N).
\end{aligned}$$

From the above replacement, the element h_i of the Heisenberg realization is replaced as following.

$$q^{h_i} \rightarrow \begin{cases} e^{a_{\pm}^i(z) + \sum_{l=1}^i (b_{\pm}^{l,i+1}(z) - b_{\pm}^{l,i}(z)) + \sum_{l=i+1}^N (b_{\pm}^{i,l}(z) - b_{\pm}^{i+1,l}(z))} & (1 \leq i \leq N-1), \\ e^{a_{\pm}^N(z) - \sum_{l=1}^{N-1} (b_{\pm}^{l,N}(z) + b_{\pm}^{l,N+1}(z))} & (i = N). \end{cases}$$

We impose q -shift to variable z of the operators $a_{\pm}^i(z)$, $b_{\pm}^{i,j}(z)$. For instance, we have to replace $a_{\pm}^i(z) \rightarrow a_{\pm}^i(q^{\pm \frac{c+N-1}{2}}z)$. Bridging the gap by the q -shift, we have the bosonizations $\psi_i^{\pm}(q^{\pm \frac{c}{2}}z) \in U_q(\widehat{sl}(N|1))$ from $q^{h_i} \in U_q(sl(N|1))$.

$$\begin{aligned}
\psi_i^{\pm}(q^{\pm \frac{c}{2}}z) &= e^{a_{\pm}^i(q^{\pm \frac{c+N-1}{2}}z) + \sum_{l=1}^i (b_{\pm}^{l,i+1}(q^{\pm(l+c-1)}z) - b_{\pm}^{l,i}(q^{\pm(l+c)}z))} \\
&\times e^{\sum_{l=i+1}^N (b_{\pm}^{i,l}(q^{\pm(c+l)}z) - b_{\pm}^{i-1,l}(q^{\pm(c+l-1)}z)) + b_{\pm}^{i,N+1}(q^{\pm(c+N)}z) - b_{\pm}^{i+1,N+1}(q^{\pm(c+N-1)}z)}, \\
\psi_N^{\pm}(q^{\pm \frac{c}{2}}z) &= e^{a_{\pm}^N(q^{\pm \frac{c+N-1}{2}}z) - \sum_{l=1}^{N-1} (b_{\pm}^{l,N}(q^{\pm(c+l)}z) + b_{\pm}^{l,N+1}(q^{\pm(c+l)}z))}.
\end{aligned}$$

In this replacement, one element q^{h_i} goes to two elements $\psi_i^{\pm}(q^{\pm \frac{c}{2}}z)$. Hence this replacement is not a map. Replacements from e_i, f_i to $x_i^{\pm}(z)$ are given by similar way, however they are more complicated. See details in [2].

3.4 Wakimoto Realization

In this section we give the Wakimoto realization $\mathcal{F}(p_a)$ whose character coincides with those of the Verma module [14]. We introduce the operators $\xi_m^{i,j}$ and $\eta_m^{i,j}$ ($1 \leq i < j \leq N, m \in \mathbf{Z}$) by

$$\eta^{i,j}(z) = \sum_{m \in \mathbf{Z}} \eta_m^{i,j} z^{-m-1} =: e^{c^{i,j}(z)} :, \quad \xi^{i,j}(z) = \sum_{m \in \mathbf{Z}} \xi_m^{i,j} z^{-m} =: e^{-c^{i,j}(z)} :.$$

The Fourier components $\eta_m^{i,j} = \oint \frac{dz}{2\pi\sqrt{-1}} z^m \eta^{i,j}(z)$, $\xi_m^{i,j} = \oint \frac{dz}{2\pi\sqrt{-1}} z^{m-1} \xi^{i,j}(z)$ ($m \in \mathbf{Z}$) are well defined on the space $F(p_a)$. We focus our attention on the operators $\eta_0^{i,j}, \xi_0^{i,j}$ satisfying $(\eta_0^{i,j})^2 = 0, (\xi_0^{i,j})^2 = 0$. They satisfy

$$\text{Im}(\eta_0^{i,j}) = \text{Ker}(\eta_0^{i,j}), \quad \text{Im}(\xi_0^{i,j}) = \text{Ker}(\xi_0^{i,j}), \quad \eta_0^{i,j} \xi_0^{i,j} + \xi_0^{i,j} \eta_0^{i,j} = 1.$$

We have a direct sum decomposition.

$$\begin{aligned}
F(p_a) &= \eta_0^{i,j} \xi_0^{i,j} F(p_a) \oplus \xi_0^{i,j} \eta_0^{i,j} F(p_a), \\
\text{Ker}(\eta_0^{i,j}) &= \eta_0^{i,j} \xi_0^{i,j} F(p_a), \quad \text{Coker}(\eta_0^{i,j}) = \xi_0^{i,j} \eta_0^{i,j} F(p_a) = F(p_a) / (\eta_0^{i,j} \xi_0^{i,j}) F(p_a).
\end{aligned}$$

We set the operator η_0, ξ_0 by

$$\eta_0 = \prod_{1 \leq i < j \leq N} \eta_0^{i,j}, \quad \xi_0 = \prod_{1 \leq i < j \leq N} \xi_0^{i,j}.$$

Definition 3.6 [14] We introduce the subspace $\mathcal{F}(p_a)$ by

$$\mathcal{F}(p_a) = \eta_0 \xi_0 F(p_a).$$

We call $\mathcal{F}(p_a)$ the Wakimoto realization.

4 Screening and Vertex Operator

In this section we give the screening that commutes with the quantum superalgebra $U_q(\widehat{sl}(N|1))$. We propose the vertex operators and the correlation functions.

4.1 Screening

In this section we give the screening \mathcal{Q}_i ($1 \leq i \leq N$) that commutes with the quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \neq -N+1$ [15]. The Jackson integral with parameter $p \in \mathbf{C}$ ($|p| < 1$) and $s \in \mathbf{C}^*$ is defined by

$$\int_0^{s\infty} f(z) d_p z = s(1-p) \sum_{m \in \mathbf{Z}} f(sp^m) p^m.$$

In order to avoid divergence we work in the Fock space.

Theorem 4.1 [15] The screening \mathcal{Q}_i commutes with the quantum superalgebra.

$$[\mathcal{Q}_i, U_q(\widehat{sl}(N|1))] = 0 \quad (1 \leq i \leq N).$$

We have introduced the screening operators \mathcal{Q}_i ($1 \leq i \leq N$) as follows.

$$\mathcal{Q}_i = \int_0^{s\infty} : e^{-(\frac{1}{k+N-1}a^i)(z)^{\frac{k+N-1}{2}}} \tilde{S}_i(z) : d_p z, \quad (p = q^{2(k+N-1)}).$$

Here we have set the bosonic operators $\tilde{S}_i(z)$ ($1 \leq i \leq N$) by

$$\begin{aligned} \tilde{S}_i(z) &= \frac{1}{(q-q^{-1})z} \sum_{j=i+1}^N : \left(e^{-b_-^{i,j}(q^{N-1-j}z) - (b+c)^{i,j}(q^{N-j}z)} - e^{-b_+^{i,j}(q^{N-1-j}z) - (b+c)^{i,j}(q^{N-j-2}z)} \right) \\ &\times e^{(b+c)^{i+1,j}(q^{N-1-j}z) + \sum_{l=j+1}^N (b_-^{i+1,l}(q^{N-l}z) - b_-^{i,l}(q^{N-l-1}z)) + b_-^{i+1,N+1}(z) - b_-^{i,N+1}(q^{-1}z)} : \\ &+ q : e^{b^{i,N+1}(z) + b_+^{i+1,N+1}(z) - b^{i+1,N+1}(qz)} : \quad (1 \leq i \leq N-1), \\ \tilde{S}_N(z) &= -q^{-1} : e^{b^{N,N+1}(z)} : . \end{aligned}$$

4.2 Vertex Operator

In this section we introduce the vertex operators $\Phi(z)$, $\Phi^*(z)$ [15]. Let \mathcal{F} and \mathcal{F}' be $U_q(\widehat{sl}(N|1))$ representation for an arbitrary level $k \neq -N+1$. Let V_α and V_α^{*S} be 2^N -dimensional typical representation

with a parameters α [21]. Let $\{v_j\}_{j=1}^{2^N}$ be the basis of V_α . Let $\{v_j^*\}_{j=1}^{2^N}$ be the dual basis of V_α^{*S} , satisfying $(v_i|v_j^*) = \delta_{i,j}$. Let $V_{\alpha,z}$ and $V_{\alpha,z}^{*S}$ be the evaluation module and its dual of the typical representation. For instance, the 8-dimensional representation $V_{\alpha,z}$ of $U_q(\widehat{sl}(3|1))$ is given by

$$\begin{aligned}
h_1 &= E_{3,3} - E_{4,4} + E_{5,5} - E_{6,6}, \\
h_2 &= E_{2,2} - E_{3,3} + E_{6,6} - E_{7,7}, \\
h_3 &= \alpha(E_{1,1} + E_{2,2}) + (\alpha + 1)(E_{3,3} + E_{4,4} + E_{5,5} + E_{6,6}) + (\alpha + 2)(E_{7,7} + E_{8,8}), \\
e_1 &= E_{3,4} + E_{5,6}, \\
e_2 &= E_{2,3} + E_{6,7}, \\
e_3 &= \sqrt{[\alpha]_q} E_{1,2} - \sqrt{[\alpha + 1]_q} (E_{3,5} + E_{4,6}) + \sqrt{[\alpha + 2]_q} E_{7,8}, \\
f_1 &= E_{4,3} + E_{6,5}, \\
f_2 &= E_{3,2} + E_{7,6}, \\
f_3 &= \sqrt{[\alpha]_q} E_{2,1} - \sqrt{[\alpha + 1]_q} (E_{5,3} + E_{6,4}) + \sqrt{[\alpha + 2]_q} E_{8,7}, \\
h_0 &= -\alpha(E_{1,1} + E_{4,4}) - (\alpha + 1)(E_{2,2} + E_{3,3} + E_{6,6} + E_{7,7}) - (\alpha + 2)(E_{5,5} + E_{8,8}), \\
e_0 &= -z(\sqrt{[\alpha]_q} E_{4,1} - \sqrt{[\alpha + 1]_q} (E_{6,2} + E_{7,3}) + \sqrt{[\alpha + 2]_q} E_{8,5}), \\
f_0 &= z^{-1}(\sqrt{[\alpha]_q} E_{1,4} - \sqrt{[\alpha + 1]_q} (E_{2,6} + E_{3,7}) + \sqrt{[\alpha + 2]_q} E_{5,8}).
\end{aligned}$$

Consider the following intertwiners of $U_q(\widehat{sl}(N|1))$ -representation [20].

$$\Phi(z) : \mathcal{F} \longrightarrow \mathcal{F}' \otimes V_{\alpha,z}, \quad \Phi^*(z) : \mathcal{F} \longrightarrow \mathcal{F}' \otimes V_{\alpha,z}^{*S}.$$

They are intertwiners in the sense that for any $x \in U_q(\widehat{sl}(N|1))$,

$$\Phi(z) \cdot x = \Delta(x) \cdot \Phi(z), \quad \Phi^*(z) \cdot x = \Delta(x) \cdot \Phi^*(z).$$

We expand the intertwining operators.

$$\Phi(z) = \sum_{j=1}^{2^N} \Phi_j(z) \otimes v_j, \quad \Phi^*(z) = \sum_{j=1}^{2^N} \Phi_j^*(z) \otimes v_j^*.$$

We set the \mathbf{Z}_2 -grading of the intertwiner be $|\Phi(z)| = |\Phi^*(z)| = 0$. For $l_a = (l_a^1, l_a^2, \dots, l_a^N) \in \mathbf{C}^N$ and $\beta \in \mathbf{C}$, we set the bosonic operator $\phi^{l_a}(z|\beta)$ by

$$\phi^{l_a}(z|\beta) =: e^{\sum_{i,j=1}^N \left(\frac{l_a^i}{k+N-1} \frac{\text{Min}(i,j)}{N-1} \frac{N-1-\text{Max}(i,j)}{1} a^j \right) (z|\beta)} :.$$

In order to balance the background charge of the vertex operators, we introduce the product of the screenings $\mathcal{Q}^{(t)}$ for $t = (t_1, t_2, \dots, t_N) \in \mathbf{N}^N$.

$$\mathcal{Q}^{(t)} = \mathcal{Q}_1^{t_1} \mathcal{Q}_2^{t_2} \dots \mathcal{Q}_N^{t_N}.$$

The screening operator $\mathcal{Q}^{(t)}$ give rise to the map,

$$\mathcal{Q}^{(t)} : \mathcal{F}(p_a) \rightarrow \mathcal{F}(p_a + \hat{t}).$$

Here $\hat{t} = (\hat{t}_1, \hat{t}_2, \dots, \hat{t}_N)$ where $\hat{t}_i = \sum_{j=1}^N A_{i,j} t_j$.

Theorem 4.2 [15] For $k = \alpha \neq 0, -1, -2, \dots, -N + 1$, bosonizations of the special components of the vertex operators $\Phi^{(t)}(z)$ and $\Phi^{*(t)}(z)$ are given by

$$\begin{aligned}\Phi_{2^N}^{(t)}(z) &= \mathcal{Q}^{(t)} \phi^{\hat{l}} \left(q^{k+N-1} z \left| -\frac{k+N-1}{2} \right. \right), \\ \Phi_1^{*(t)}(z) &= \mathcal{Q}^{(t)} \phi^{\hat{l}^*} \left(q^k z \left| -\frac{k+N-1}{2} \right. \right),\end{aligned}$$

where we have used $\hat{l} = -(0, \dots, 0, \alpha + N - 1)$, $\hat{l}^* = (0, \dots, 0, \alpha)$ and $t = (t_1, t_2, \dots, t_N) \in \mathbf{N}^N$. The other components $\Phi_j^{(t)}(z)$ and $\Phi_j^{*(t)}(z)$ ($1 \leq j \leq 2^N$) are determined by the intertwining property and are represented by multiple contour integrals of Drinfeld currents and the special components $\Phi_{2^N}^{(t)}(z)$ and $\Phi_1^{*(t)}(z)$. We have checked this theorem for $N = 2, 3, 4$.

Here we give additional explanation on the above theorem. The explicit formulae of the intertwining properties $\Phi^{(t)}(z) \cdot x = \Delta(x) \cdot \Phi^{(t)}(z)$ for $U_q(\widehat{sl}(3|1))$ are summarized as follows. We have set the \mathbf{Z}_2 -grading of V_α as follows : $|v_1| = |v_5| = |v_6| = |v_7| = 0$, and $|v_2| = |v_3| = |v_4| = |v_8| = 1$.

$$\begin{aligned}\Phi_3^{(t)}(z) &= [\Phi_4^{(t)}(z), f_1]_q, \quad \Phi_5^{(t)}(z) = [\Phi_6^{(t)}(z), f_1]_q, \\ \Phi_2^{(t)}(z) &= [\Phi_3^{(t)}(z), f_2]_q, \quad \Phi_6^{(t)}(z) = [\Phi_7^{(t)}(z), f_2]_q, \\ \Phi_1^{(t)}(z) &= \frac{1}{\sqrt{[\alpha]_q}} [\Phi_2^{(t)}(z), f_3]_{q^{-\alpha}}, \quad \Phi_3^{(t)}(z) = \frac{-1}{\sqrt{[\alpha+1]_q}} [\Phi_5^{(t)}(z), f_3]_{q^{-\alpha-1}}, \\ \Phi_4^{(t)}(z) &= \frac{-1}{\sqrt{[\alpha+1]_q}} [\Phi_6^{(t)}(z), f_3]_{q^{-\alpha-1}}, \quad \Phi_7^{(t)}(z) = \frac{1}{\sqrt{[\alpha+2]_q}} [\Phi_8^{(t)}(z), f_3]_{q^{-\alpha-2}}.\end{aligned}$$

The elements f_j are written by contour integral of the Drinfeld current $f_j = \oint \frac{dw}{2\pi\sqrt{-1}} x_j^-(w)$. Hence the components $\Phi_j^{(t)}$ ($1 \leq j \leq 8$) are represented by multiple contour integrals of Drinfeld currents $x_j^-(w)$ ($1 \leq j \leq 3$) and the special component $\Phi_8^{(t)}(z)$.

4.3 Correlation Function

In this section we study the correlation function as an application of the vertex operators. We study non-vanishing property of the correlation function which is defined to be the trace of the vertex operators over the Wakimoto module of $U_q(\widehat{sl}(N|1))$. We propose the q -Virasoro operator L_0 for $k = \alpha \neq -N + 1$ as follows.

$$\begin{aligned}L_0 &= \frac{1}{2} \sum_{i,j=1}^N \sum_{m \in \mathbf{Z}} : a_{-m}^i \frac{m^2 [\text{Min}(i,j)m]_q [(N-1 - \text{Max}(i,j))m]_q}{[m]_q [(k+N-1)m]_q [(N-1)m]_q [m]_q} a_m^j : \\ &\quad + \sum_{i,j=1}^N \frac{\text{Min}(i,j)(N-1 - \text{Max}(i,j))}{(k+N-1)(N-1)} a_0^j \\ &\quad - \frac{1}{2} \sum_{1 \leq i < j \leq N} \sum_{m \in \mathbf{Z}} : b_{-m}^{i,j} \frac{m^2}{[m]_q^2} b_m^{i,j} : + \frac{1}{2} \sum_{1 \leq i < j \leq N} \sum_{m \in \mathbf{Z}} : c_{-m}^{i,j} \frac{m^2}{[m]_q^2} c_m^{i,j} : \\ &\quad + \frac{1}{2} \sum_{1 \leq i \leq N} \sum_{m \in \mathbf{Z}} : b_{-m}^{i,N+1} \frac{m^2}{[m]_q^2} b_m^{i,N+1} : + \frac{1}{2} \sum_{1 \leq i \leq N} b_0^{i,N+1}.\end{aligned}$$

The L_0 eigenvalue of $|l_a, 0, 0\rangle$ is $\frac{1}{2(k+N-1)}(\bar{\lambda}|\bar{\lambda} + 2\bar{\rho})$, where $\bar{\rho} = \sum_{i=1}^N \bar{\Lambda}_i$ and $\bar{\lambda} = \sum_{i=1}^N l_a^i \bar{\Lambda}_i$.

Theorem 4.3 [15] *For $k = \alpha \neq 0, -1, -2, \dots, -N+1$, the correlation function of the vertex operators,*

$$\mathrm{Tr}_{\mathcal{F}(l_a)} \left(q^{L_0} \Phi_{i_1}^{*(y_{(1)})}(w_1) \cdots \Phi_{i_m}^{*(y_{(m)})}(w_m) \Phi_{j_1}^{(x_{(1)})}(z_1) \cdots \Phi_{j_n}^{(x_{(n)})}(z_n) \right) \neq 0,$$

if and only if $x_{(s)} = (x_{(s),1}, x_{(s),2}, \dots, x_{(s),N}) \in \mathbf{N}^N$ ($1 \leq s \leq n$) and $y_{(s)} = (y_{(s),1}, y_{(s),2}, \dots, y_{(s),N}) \in \mathbf{N}^N$ ($1 \leq s \leq m$) satisfy the following condition.

$$\sum_{s=1}^n x_{(s),i} + \sum_{s=1}^m y_{(s),i} = \frac{(n-m)i}{N-1} \alpha + n \cdot i \quad (1 \leq i \leq N).$$

Acknowledgements.

This work is supported by the Grant-in-Aid for Scientific Research **C** (21540228) from Japan Society for Promotion of Science. The author would like to thank Professor Branko Dragovich and the organizing committee of the 7-th Mathematical Physics Meeting for an invitation.

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